George W. Bush Institute

Middle School Matters Field Guide:

Research-Based Principles, Practices, and Tools

Chapter 1: Research-Based Instruction

Mathematics and Mathematics Interventions

GEORGE W. BUSH INSTITUTE

The Meadows Center FOR PREVENTING EDUCATIONAL RISK

Preferred Citation

The Meadows Center for Preventing Educational Risk & George W. Bush Institute. (2016). *Middle School Matters field guide: Research-based principles, practices, and tools* (2nd ed.). Austin, TX: Authors.

Mathematics and Mathematics Interventions

Mathematics includes knowledge and skills that are used to access science, technology, engineering, and mathematics applications and careers and that improve reasoning and analytic thinking. Given the importance of algebra for success in high school and beyond, many states have moved to increase student expectations in this area.[1] Thirty-four states now require Algebra I or its equivalent for graduation.[1] Despite the effort to increase expectations, achievement data indicate students continue to struggle to attain proficiency, especially those with disabilities and difficulties. As a result, the National Mathematics Advisory Panel[3] convened, and a careful examination of the research evaluating the effectiveness of instructional practices for struggling learners was launched. The following nine key principles reflect research in mathematics education synthesized by the Panel and the Institute of Education Sciences.[2, 3, 4, 5]

Principle 1:

Establish school wide practices for enhancing mathematics understanding within relevant content area instruction.

Mathematics is a foundational tool that can be used across many content areas (e.g., integrated science, social studies, health). Doing so provides students with important opportunities to practice and apply mathematical tools and skills. Therefore, it is important that content area teachers consider the mathematics relevant to their discipline and expect students to be able to use their mathematical knowledge and skills to better understand and communicate what they are learning in other content domains.

Practice 1: *Encourage students to apply their understanding of mathematics concepts and procedures to draw conclusions and propose solutions about history, science, social studies, economics, and other content areas.*

In many cases, content area teachers will find instances in which mathematics can be used to summarize, illustrate, explain, or analyze information. Reference materials, textbooks, and primary source documents frequently include related information that can be displayed graphically; summarized using statistics, including measures of central tendency; and/or interpreted to draw conclusions and make predictions.

Practice 2: *Ask students to analyze events and phenomena from a quantitative perspective and use their analyses to develop arguments and provide justifications.*

Many content disciplines (e.g., history, science, mathematics) are taught in an isolated fashion but are related in important ways. For example, many historical events can be described as a human response to natural phenomena (science) and can be summarized and analyzed using data (mathematics). While reading and writing help with acquisition and comprehension, mathematics and quantitative reasoning (e.g., economic analysis, historical trends, epidemiological analysis) can help with interpretation and broadening perspectives.

PRACTICE 1 EXAMPLE APPLICATION: Apply Mathematics in Science

A physical science teacher will find daily opportunities for students to collect and analyze data based on their observations. For example, students may be studying objects, such as marbles or toy cars, moving at a fixed distance. They can record the time each object takes to move the distance and from that information determine the speed as a function of distance and time. These observations can be organized in tables and graphics, and conclusions can be drawn about the objects' features that result in differences in speed. This type of mathematics application can be expanded further to include proportional reasoning by having students examine how the speed changes with varying distances.

PRACTICE 2 EXAMPLE APPLICATION: Quantitative Reasoning in Social Studies

Quantitative reasoning can be used in teaching social studies by having students examine sources regarding population changes, technological advances, and timelines to draw conclusions about the factors (e.g., warfare, migration, etc.) that contributed to specific historic events. For example, a social studies teacher might introduce students to the spice trade. Students today are likely to struggle to understand why spices were so valuable in the fourteenth century. However, when presented with data comparing the value of today's common spices with their value to Europeans in the 1300s, students will quickly understand why people risked their lives to search for new markets. Students can use their quantitative reasoning to explore predictions for similar risks people take today.

Principle 2:

Use a universal screener to identify students at risk for mathematics difficulties and to determine interventions to provide these at-risk students. Monitor the development of mathematics knowledge and skills of identified students.[4]

The universal screener[a] ensures that schools can identify students who will likely need support beyond the typical core instruction provided in the general education classroom. Subsequent progress monitoring should be used to continue to track the effectiveness of interventions. The frequency of progress monitoring should reflect the level of support needed, with more vigilant monitoring used for students with greater needs.

Practice 1: *Identify a system for screening and progress monitoring that prioritizes content and skills necessary for subsequent mathematics development.*

The content of the universal screener should reflect mathematics knowledge and skills essential for grade-level proficiency. The specific content selected should relate to the domain in which potential risk is being evaluated. For example, if the assessment system is screening for potential risk in algebra and algebra readiness skills and knowledge, the universal screener should target content knowledge and skills related to algebra and algebra readiness.

a A universal screening is a quick assessment that allows for repeated testing of all students on skills of interest. This kind of screening is used to determine the efficacy of the curriculum and instruction and the level of student proficiency in a specific area (e.g., mathematics, reading). Then school administrators and teachers can analyze the data by group, by individual student, and by specific skill to make decisions about interventions to improve student outcomes. These screeners are typically given to all students at three time points in a year.

Practice 2: *Select a cut score for screening that balances the need to help the most at-risk students with the resources available.*

In a universal screening system, cut scores are used to identify ranges of scores along the distribution that might indicate risk status. For example, three cut-scores could be established to identify students who might be at risk for failure and the significance of the risk. In this model, the distribution of scores could be divided into three categories to indicate students who might have minimal risk of failure, students who have some risk of failure, and students who have significant risk of failure.

A number of factors should be considered in determining the appropriate cut score used in screening. Ensuring that children who need additional support are accurately identified and that resources are available to provide the additional support are two essential considerations.

Principle 3:

Help students recognize number systems and expand their understanding beyond whole numbers to integers and rational numbers. Use number lines as a central representational tool in teaching this and other rational number concepts.[4, 6]

There is growing understanding that students' preparation for success in algebra lies in their proficiency with numeracy.[6] Numeracy, often called "number sense," refers to a "child's fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons."[7] Numeracy includes students' conceptual understanding of number systems and of the properties and operations that govern lawful application during problem solving. Numeracy also relates to students' ability to extend generalizations about one number system to another. By generalizing, students can apply their knowledge of concrete numbers to algebraic reasoning.

Practice 1: *Use measurement activities and number lines to help students understand that fractions and decimals are numbers and share number properties.[8]*

Using number lines is an effective way to illustrate that fractions and decimals have magnitude similar to whole numbers. Measuring objects using number lines with fractions demonstrates to students that fractions have magnitude. For example, the following illustration (Figure 1) shows how a simple number line with fractions can be used to measure a common object.

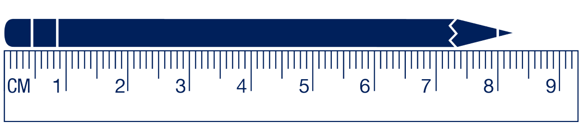


FIGURE 1. Measuring common objects with fractions

Practice 2: *Provide opportunities for students to locate and compare fractions and decimals on number lines.*

By accurately placing fractions and decimals on a number line, students learn to make relative comparisons. Comparing fractions and decimals facilitates students' understanding of the role magnitude plays in increasing precision of measurement.[9] Relative comparisons, as such, also help students justify the reasonability of their answers. A sample number line for comparing fractions is provided in Figure 2 within the *Practice 2 Example Application.*

Practice 3: *Use number lines to improve students' understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions.*

Comparing and extending number lines uses visual representations to develop students' understanding of equivalent fractions. Placing increasingly smaller fractions on a number line illustrates a unique characteristic of rational numbers: They can represent increasingly precise values between whole numbers. A sample number line is provided in Figure 3.

PRACTICE 2 EXAMPLE APPLICATION: Sample Lesson for Comparing Fractions

Objectives: Students will be able to connect fractions to a number line and use a number line to identify equivalent fractions.

Instruction: Compare fractions on the number lines such as those in Figure 2. Select one pair of equivalent fractions (e.g., $\frac{3}{4}$ and $\frac{6}{8}$) and explain to students that these fractions are equivalent because you can multiply the numerator and denominator of one fraction by the same number (e.g., 2) to get the other fraction. Point out that when fractions are equivalent, they are located on the same place on the number line.

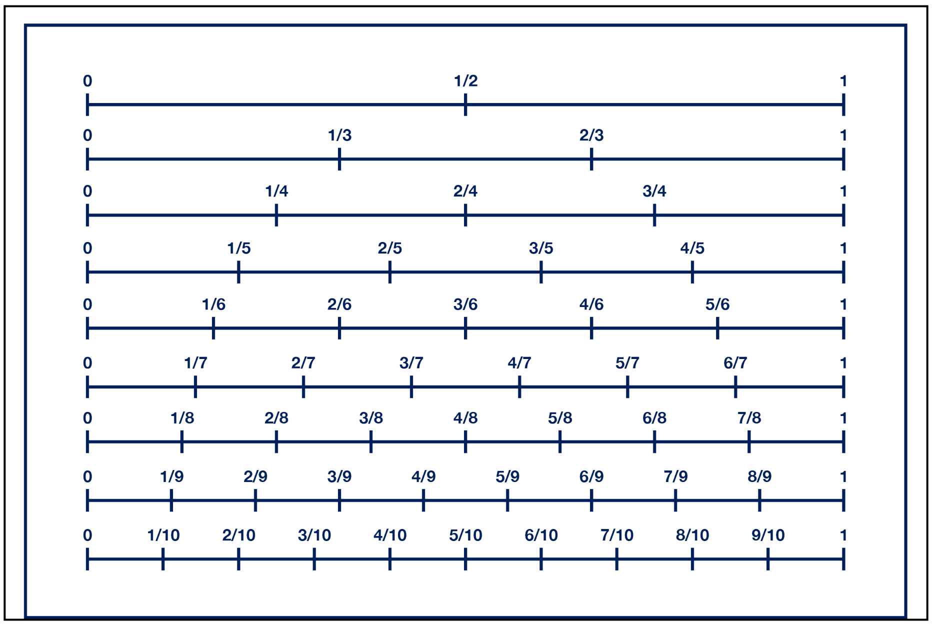


FIGURE 2. Number Line Representations for Comparing Fractions

Practice 4: *Explain that fractions can be represented as common fractions, decimals, and percentages, and develop students' ability to translate among these forms.*

Students must learn that numbers are representations of values and that these representations can be expressed in different forms. Students need to be able to transition between these different forms efficiently. For example, students should understand that

$$\frac{8}{10} = 0.80 = 80\%$$

Just as number lines can be used to develop an understanding of equivalent fractions, they can also be used to demonstrate equivalence among representations of rational numbers. For example, the illustration in Figure 3 demonstrates how a number line can be used to emphasize equivalence between fractional representations and decimal representations, such as

$$0.4 = \frac{4}{10} = \frac{2}{5} \qquad 0.4 < \frac{1}{2}$$

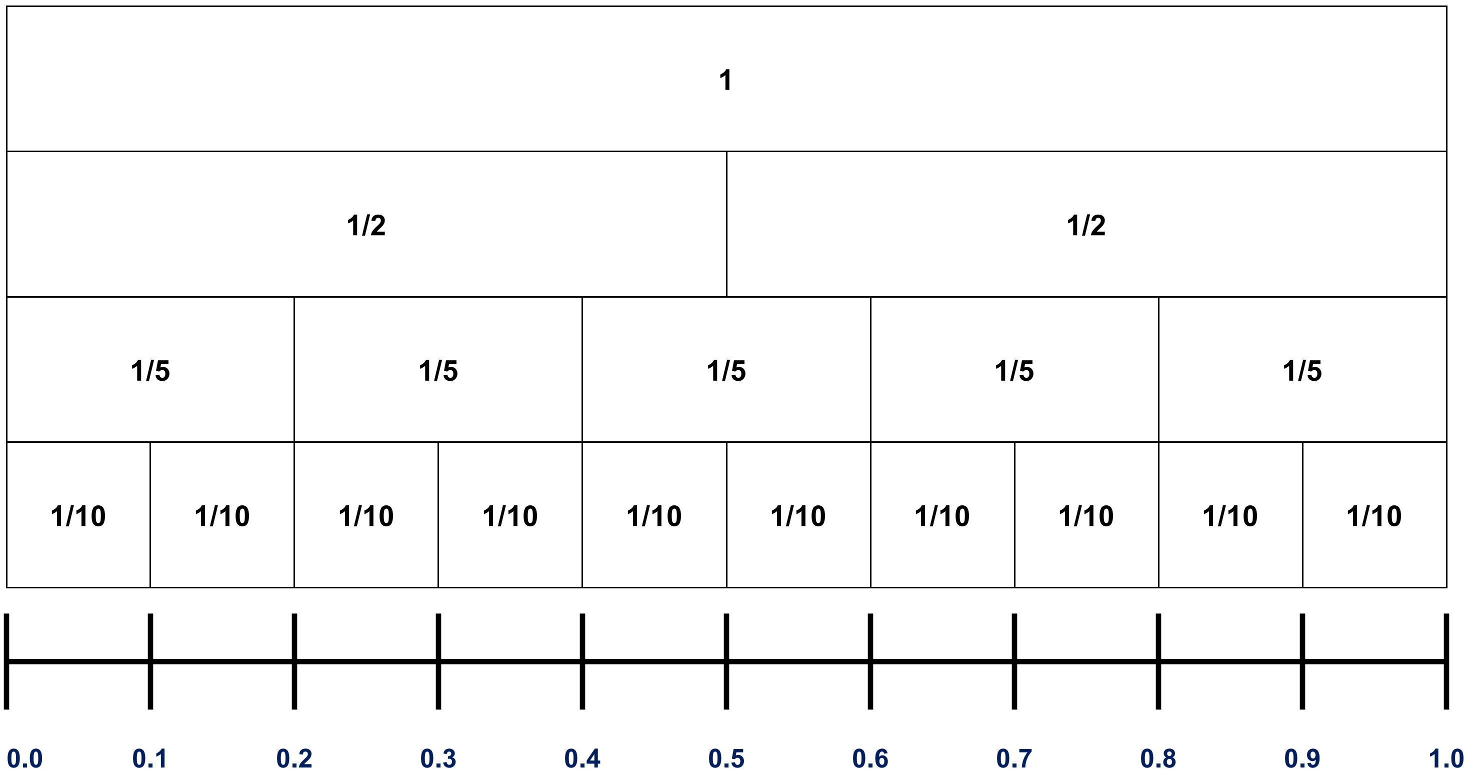


FIGURE 3. Number Line Representation of Equivalent Fractions

Principle 4:

Develop students' conceptual understanding of mathematics and provide ample opportunities to improve procedural fluency.[3, 4, 10]

Ultimately, the goal is to improve students' abilities to apply mathematics knowledge to solve problems. Evidence suggests that overall outcomes are improved when students develop conceptual understanding of mathematics in addition to procedural fluency.[11] A balanced approach to instruction helps students understand why procedures for computations with fractions and other number systems and algebraic expressions make sense. For students receiving intervention, specific amounts of allocated time should be spent on improving procedural fluency.

Practice 1: *Use area models, number lines, and other visual representations to improve students' understanding of formal computational procedures.*

Students' application of mathematical procedures should be based on a strong conceptual foundation. This foundation can be built by demonstrating mathematical procedures through physical and representational models. For example, the illustration in Figure 4 can be used to help students understand the need to find a common denominator when adding fractions.

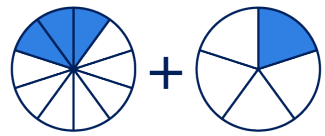


FIGURE 4. Representation for Finding Common Denominators

Resources for implementing this practice can be found at the Middle School Matters Institute website:

- **Visual Representations Toolkit**
  https://greatmiddleschools.org/toolkits/math/visual-representations/

Practice 2: *Use meaningful fact practice activities for students lacking a strong foundation in math facts.*

Practice, not drill, is necessary to learn any new skill. This is especially true in mathematics. To aid in computational efficiency, students need comprehensive knowledge and recall of math facts for addition, subtraction, multiplication, and division. To begin, teachers should assess the fluency of students. If students experience difficulty with fluency, teachers should explicitly teach early numeracy and operations concepts. Daily practice activities, conducted for a brief amount of time, can improve math fact fluency. Technology can also be used for math fact practice.

Practice 3: *Address common misconceptions regarding computational procedures.*

Students' errors can provide valuable information to guide instruction. Often, students generally understand the procedure but inadvertently make mistakes when executing computations. However, in some instances, these mistakes represent persistent misconceptions or misapplications. Through his research, Ashlock identified three basic categories of computational errors: (a) wrong operation, in which the student uses an inappropriate operation to solve a problem, (b) computational error, in which the student uses the appropriate operation but makes an error involving basic number facts, and (c) defective algorithm, in which the student uses the appropriate operation but makes a non-number fact error in one or more steps of the operation.[12] Defective algorithms often result when students over-generalize procedures without relying on standard algorithms or a foundational understanding of the procedure, as is illustrated in the *Practice 3 Example Application*. These misconceptions need to be addressed in instruction to prevent chronic errors.

In terms of algorithms, students should learn a standard algorithm for each distinct type of computation. A standard algorithm is a set of steps that will work for a type of computation problem (e.g., multi-digit multiplication). Teachers can expose students to other strategies to provide different options for approaching novel problems.[1] However, students should not be forced to demonstrate fluency with all strategies.

Practice 4: *Present real-world contexts with plausible numbers for problems.*

Students often find solving problems that are situated in real-world examples to be motivating and relevant to their experiences. Teachers should take care, however, to use real-world contexts that are meaningful while still maintaining the intended mathematical ideas.[13]

PRACTICE 3 EXAMPLE APPLICATION: Proportional Reasoning

In proportional reasoning, teachers commonly teach students the strategy of cross-multiplying. Students become proficient at the procedure but do not understand why cross multiplication is a lawful, efficient way to solve a problem involving a proportion.

$$\frac{2}{3} = \frac{x}{6}$$

When cross-multiplied, this is $3x = 12,$ and $x = 4$.

However, if students understand that equivalent fractions represent the same magnitude and, by definition, they can multiply both sides by the same number and the fractions remain equivalent. Therefore, if students multiply both sides by the product of the denominators the result is:

$$\frac{2}{3}(18) = \frac{x}{6}(18)$$

When simplified, this is:

$$\frac{36}{3} = \frac{18x}{6}$$

$12 = 3x$, and $x = 4$

Cross multiplication is efficient, but unless students understand the underlying mathematics, it could lead to errors and possible misconceptions.

Principle 5:

Provide explicit and systematic instruction during instruction and intervention.[3]

In core instruction, teachers often engage students in learning about a new topic or procedure by having them explore the mathematics teachers believe will be involved. Due to the large teacher-student ratio, the guidance during practice, the corrective feedback during instruction, and the review may be insufficient to help struggling students make progress toward expected outcomes. Students having difficulty with mathematics require explicit and systematic instruction that includes modeling concepts, procedures, and proficient problem solving processes; verbalizing thought processes; guided practice; corrective feedback; and frequent cumulative review.[14]

Practice 1: *Include explicit teacher or peer modeling and demonstrate key concepts and skills.*

Students who struggle in mathematics need an instructional model from a teacher or a more proficient peer that is clearly communicated and that illustrates the steps in the mathematical procedure so the learner can repeat them successfully.[15, 16] The teacher should use precise language to present clear models, and these models should include examples to illustrate concepts and demonstrate procedures. By verbalizing the thinking involved in completing the model, the teacher helps students develop the critical thinking skills and the vocabulary that will guide them through similar problems in the future.[17] As students develop understanding, instruction will expand to a broader range of examples that may increase in complexity.

Resources for implementing this practice can be found at the Middle School Matters Institute website:

- **Peer-Mediated Instruction Toolkit**
  https://greatmiddleschools.org/toolkits/math/peer-mediated-instruction/

Practice 2: *Include worked examples of key concepts and skills.* [1]

A teacher's or peer's modeling may include worked examples that the teacher or peer analyzes and discusses in the context of the step-by-step algorithm or process used to work the examples. For example, in the *Practice 1 Example Application*, a teacher could present the work (written in the left column) and use the think-aloud presented in the right column.

Practice 3: *Gradually transition from teacher-modeled problem solving to student-directed problem solving.*

Students need to develop independent problem solving strategies. It is helpful to provide students with a framework for problem solving, such as a step-by-step checklist or mnemonic that reminds students of the problem-solving process. This process can involve coaching and prompting that decreases as students become more proficient.

PRACTICE 1 EXAMPLE APPLICATION: Proportional Reasoning Using a Think-Aloud Strategy

In the following example, a teacher provides students with explicit instruction on how to change a mixed number to an equivalent fraction.

Example	Teacher Think Aloud
$2\frac{2}{3}$	How can I convert this to another fraction with the same value?
$= 1 + 1 + \frac{2}{3}$	First we convert the whole number 2 to two ones. How can I add whole numbers to the fraction? I have to also make them fractions.
$= \frac{3}{3} + \frac{3}{3} + \frac{2}{3}$	Next, we convert the whole numbers to fractions with the same denominator as two thirds.
$= \frac{8}{3}$	Can I add these fractions? Yes, they have the same denominator, so I add the numerators. 3 plus 3 plus 2 equals 8. The answer is eight thirds.

PRACTICE 3 EXAMPLE APPLICATION: Transitioning from Teacher-Modeled Problem Solving to Student-Directed Problem Solving

In the example below, the teacher uses questions to guide students thinking through a proportional reasoning problem:

For every 30 students at school, 10 bring their lunch. If there are 700 students in the school, how many total students bring their lunch to the cafeteria?

- Let's highlight the key information. What two quantities are being compared? Highlight those two numbers.
- From this situation, can you write a proportion comparing two ratios? Why would it be important to include the units that are being compared? (To make sure that the proportion is comparing the same ratios)
- What is this problem asking? (How many students bring their lunch to the cafeteria if there are 700 students in the school?)
- Now, let's plan how to solve this problem. What quantities am I comparing? (Number of students bringing lunch to the cafeteria and total number of students)
- What do I know? (10 out of 30 students bring their lunch; there are 700 students in the school.)
- What quantities go together? (10 students bring their lunch for every 30 students; x students bring their and 700 total students in the school.)
- What am I looking for? (The number of students who bring their lunch out of 700 total students)
- How would I set this up?

$$\frac{\text{students eat in cafeteria}}{\text{total students}} = \frac{\text{students eat in cafeteria}}{\text{total students}}$$

Practice 4: *Include opportunities for students to talk aloud about the skills, knowledge, or problem-solving procedures they are learning.*

Encouraging students to verbalize the procedures for problem-solving and their rationale for each step will increase awareness of their thinking. This verbalization may initially need to be modeled by the teacher or a peer.[16, 17] (See the *Practice 4 Example Application*)

Practice 5: *Provide immediate and corrective feedback with opportunities for students to correct errors.*

Struggling learners are often unaware of their mistakes. Immediate and corrective feedback clarifies their understanding and reinforces accuracy. In the following example, a student is using multiplication to find equivalent fractions. (See the *Practice 5 Example Application*)

Practice 6: *Include sufficient, distributed, and cumulative practice and review.*

Many struggling learners need sufficient practice and review to develop mastery. Often, students remember more when they are exposed to the new information on two occasions (instead of just one) and when there is time between those two occasions.[18] This practice is called *delayed review*.[19] In addition, by distributing types of problems across assignments, students can experience overlearning of these topics or skills, and students learn to discriminate among operations and strategies.[20] An example would be to include a few review problems on homework and classwork each night for a few weeks and then once a week after that. This practice is associated with improved retention.

PRACTICE 4 EXAMPLE APPLICATION: Encouraging Students to Talk About their Thinking

Situation
You buy 2 packages of pencils for $4. How much would it cost to buy 12 packages?
Teacher: I see that you use different ways to find the answer. Mark, can you tell us how you solved the problem? **Mark**: Sure. I knew that if I could figure how much 1 package costs, then I could multiply it by 12 to get the cost of 12 packages. You can buy 2 packages for $4, so you know 1 package is $2. So, I can multiply $2 by 12 to get $24 for 12 packages. **Teacher**: Mark, how did you find out that 1 package is $2? **Mark**: Oh, since 2 packages was $4, I divided $4 by 2, to get the cost of 1 package, $2. **Teacher**: Okay, and do you remember what we called it when we found the cost of 1 package? **Mark**: The unit rate? **Teacher**: Yes! The unit rate is the cost for one package.

PRACTICE 5 EXAMPLE APPLICATION: Immediate and Corrective Feedback

Example of Reva's work

$$\frac{2}{3}(2) = \frac{4}{3}(2) = \frac{8}{3}(2) = \frac{16}{3}$$

Teacher: Reva, explain how you found these equivalent fractions and how you know they represent the same magnitude.

Reva: Well, I multiplied each one by the same number, 2, to get the next fraction.

Teacher: What is the only number that we can use as a multiplier to get an equivalent fraction?

Reva: 1.

Teacher: That's right. Right now you are multiplying each fraction by $\frac{2}{1}$.

What would you have to multiply each fraction by to get an equivalent fraction and still use 2 as the multiplier in your numerator?

Reva: $\frac{2}{2}$?

Teacher: That's right! Let's go back and use $\frac{2}{2}$.

Principle 6:

Instruction should include strategies for solving word and algebra problems that are based on common underlying structures.[17]

For Word Problems:
Common word-problem structures have growing support in literature and have been successfully employed in mathematics instruction for some time.[21] Intervention programs are increasingly focusing on these structures in an effort to help students identify the relevant information in a problem. From this information provided in the structure, students can systematically identify an effective approach to solving the problem.

Typical word-problem structures for additive word problems include combine, compare, and change. Structures to solve multiplicative word problems include equal groups, comparisons, and combinations. While these problem structures should be a focus of instruction in the elementary grades, middle grades students may need a review and additional practice solving word problems with these structures. Often, at the middle grades, students solve multi-step additive and multiplicative word problems that include whole or rational numbers and feature information from tables, charts, or graphs.

In the middle grades, a common problem structure relates to ratios. Problems that involve ratios generally include two problems that are being compared and ask the student to use that comparison to find a missing number.[22] The problem below illustrates this:

Problem:
Each time Jim inserts 2 tokens into the gumball machine he gets 3 gumballs. How many gumballs would he get if he puts in 20 tokens?

The numbers being compared are 2 tokens and 3 gumballs. The missing number is the number of gumballs he would receive if he put in 20 tokens. This problem structure is common to ratio problems. After students work several problems of this type and learn how to identify the common structure, they are able to categorize this as a ratio problem.

For Algebra:
Teaching students to focus on problem "structure helps students make connections among problems, solution strategies, and representations that may initially appear different but are actually mathematically similar."[1]

For example, the equations $2x + 8 = 14$, $2(x + 1) + 8 = 14$, and $2(3x + 4) + 8 = 14$ may look different, but to solve each equation 2 is multiplied by a quantity and 8 is added to sum to 14.

Practice 1: *Include systematic instruction on the structural connections between known, familiar, and novel word problems.*

Teach an organizational strategy for setting up and solving problems. Then, help students identify underlying structures of problems across a range of examples to ensure generalization. Students who receive instruction in finding the underlying structures in problems are able to solve problems more efficiently because they have a schema for the problem and do not have to relearn how to set it up with each new problem they encounter. They also are better able to recognize meaningful features of a problem, such as the following example problem, and do not rely on key words or superficial features of the context.

Examine the *Practice 1 Example Application*. Students who have been explicitly taught how to identify the features of a ratio problem rather than seeing its superficial structure will see that there are two compared numbers in each of these problems. However, in the third problem, students would look at the question and would determine that it is a problem that doesn't require a ratio.

PRACTICE 1 EXAMPLE APPLICATION: Proportional Reasoning Word Problems

Problem: Felicity rides her bike at a rate of 4 miles per hour. How many hours will it take her to ride 22 miles at a constant rate?

Problem: The gas mileage for Elijah's car is 22 miles per gallon. How many gallons of gas would it take to drive 432 miles at a constant rate?

Problem: Each time Susan's mom deposits $100 in Susan's savings account, she also deposits $50 in Susan's brother's account. How much money does she have to deposit each time?

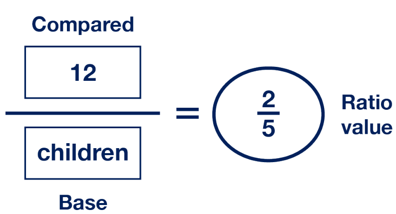
Practice 2: *Teach common problem types and their structures, as well as how to categorize and select appropriate solution methods for each problem type.*

Proficiency with problem solving depends on the student's ability to see common problem types and connect them to viable solutions. Struggling students need to receive explicit instruction on organizing information presented in word problems, on common problem types, and appropriate solutions.[23, 21]

Resources for implementing this practice can be found at the Middle School Matters Institute website:

- **Schema-Based Instruction Toolkit:** https://greatmiddleschools.org/toolkits/math/schema-based-instruction/

PRACTICE 2 EXAMPLE APPLICATION: Teaching Common Problem Types—Ratio Problems

Problem	Teacher Language
The ratio of the number of girls to the total number of children in Ms. Robinson's class is 2:5. The number of girls in the class is 12. How many children are in the class?	**Step 1:** Find the problem type. • Read and retell problem to understand it. • Ask if the problem is similar or different from other problems. • Is this a ratio problem?
Ratio Problem 	**Step 2:** Organize the problem. • The number of girls in the class is 12. • The ratio of the number of girls to the total number of children in Ms. Robinson's class is 2:5.
$\frac{12\ girls}{d\ children} = \frac{2}{5}$	**Step 3:** Plan to solve the problem.
$\frac{12\ girls(5d)}{d\ children} = \frac{2\ (5d)}{5}$ $60\ = 2d$ $30\ children = d$	**Step 4:** Solve the problem. • Multiply both sides by the product of the denominators. • Simplify and multiply both sides. • Divide both sides by 2.

Principle 7:

For students who struggle in mathematics, instruction and intervention materials should include opportunities to work with representations of mathematical ideas.[24] Teachers should be proficient in the use of these representations.

As discussed in Principle 4, Practice 1, evidence suggests that using visual representations during instruction can improve students' conceptual understanding and procedural fluency. For students who are struggling, research indicates that using visual representations during lessons may improve students' overall outcomes in mathematics, pre-algebra proficiency, problem-solving skills, and procedural fluency. Many mathematics textbooks provide little information about using visual representations, so teachers should use explicit instruction to introduce and explain all visual representations to students.[25] Through carefully sequenced instruction, visual representations appear to help students transition from concrete models of mathematical ideas to abstract mathematical representations. Moreover, students can use visual representations to develop strategies for translating word problems into abstract numerical statements.

Practice 1: *Employ visual representations to model mathematical concepts.*

Foundational understanding of mathematics depends on seeing its application to physical and visual models. Intervention instruction should incorporate concrete and visual (i.e., semi-concrete) representations of mathematical concepts to develop foundational knowledge. Instruction, however, should use representations as a support for mathematics learning rather than a focus of the lesson.[26]

Practice 2: *Explicitly link a visual representation or model with the abstract mathematical symbol or concept.*

To learn to interpret and comprehend abstract mathematics, students must see and understand how a visual representation can be translated into abstract numbers and number sentences. See Figures 1, 2, 3, and 4 for visual representations related to fractions. See the *Practice 2 Example Application* on the following page for an example related to algebra.

Practice 3: *Use consistent language across similar representations.*

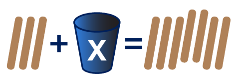
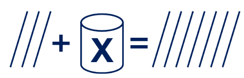
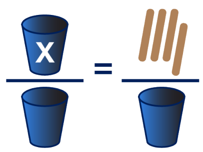
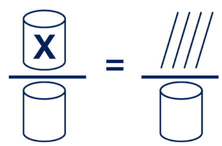
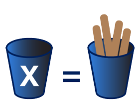
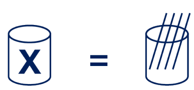
Students will be better able to perceive similarities across representations if language is consistent and precise. For example, interventionists should use the same mathematically precise language when using concrete models, visual representations, and abstract mathematical notation. (See *Practice 3 Example Application*)

Resources for implementing this practice can be found online:

- **Visual Representations Toolkit**
  https://greatmiddleschools.org/toolkits/math/visual-representations/

PRACTICE 2 EXAMPLE APPLICATION: Matched Concrete, Visual, and Abstract Representations

3 + X = 7

Solving the Equation with Concrete Manipulatives (Cups and Sticks)	**Solving the Equation with Visual Representations of Cups and Sticks**	**Solving the Equation with Abstract Symbols**
A 		**3 + 1X = 7**
B 		$\underline{-3 \qquad -3}$
C **D** 		$\frac{1X}{1} = \frac{4}{1}$
E 		**X = 4**

Concrete Steps
A. 3 sticks plus one group of X equals 7 sticks
B. Subtract 3 sticks from each side of the equation
C. The equation now reads as one group of X equals 4 sticks
D. Divide each side of the equation by one group
E. One group of X is equal to four sticks (i.e., 1X/group = 4 sticks/group; 1X = 4 sticks)

PRACTICE 3 EXAMPLE APPLICATION: Consistent and Precise Language

In the *Practice 2 Example Application,* you can see how solving for an unknown number in an equation is displayed using concrete examples, visual representations, and numerals. In all three forms, the language should be consistent.

For example, in Step A, the teacher and students can say: “Three sticks plus one group of some number of sticks is seven sticks.”

Step B: “Subtract three sticks from both sides.”

Step C: “One group of unknown number sticks is four sticks.”

Step D: “Divide each side of the equation by one group.”

Step E: “The unknown number sticks is four sticks.”

Principle 8:

Establish a school wide plan to identify and improve teachers' mathematical and pedagogical content knowledge.

Mathematical content knowledge refers to teachers' understanding of the precise mathematics that underlie the content being taught. "Teachers must know in detail the mathematical content they are responsible for teaching and its connections to other important mathematics, both prior to and beyond the level they are assigned to teach."[3] Teachers with a deep understanding of mathematics possess a conceptual understanding of the entire elementary and middle grades mathematics curriculum that goes beyond simply being able to execute the algorithms.[27]

Pedagogical content knowledge refers to the integration of mathematical content knowledge with knowledge of students, learning, and the influences of pedagogy on instructional practice.[28] Pedagogical content knowledge should inform the design and delivery of instruction, as well as interactions with students during the learning process. Specifically, teachers' pedagogical content knowledge is elicited when reasoning through mathematical explanations that arise in response to students' questions, analyzing textbooks and instructional aids for appropriateness and/or alignment with instructional objectives, and creating instructional materials, such as tests and homework exercises.

The research demonstrating the impact of teachers' mathematical and pedagogical content knowledge is limited and often is not designed to evaluate causal relationships. However, numerous mathematics education researchers and mathematicians hypothesize that teachers' mathematical and pedagogical content knowledge is highly correlated with student achievement.[3] Wu notes the impact on student engagement stating, "teachers who can make transparent what they are talking about (cf. definitions and precision), can explain what they ask students to learn (cf. reasoning and coherence), and can explain why students should learn it (cf. purposefulness) have a much better chance of opening up a dialogue with their students and inspiring them to participate in the doing of mathematics" (p. 6-7). [29]

Practice 1: *Assess teachers' needs in relation to mathematics content knowledge and mathematics pedagogical content knowledge across content areas.*

To accurately target professional development, mathematics teachers' depth and breadth of mathematical and pedagogical content knowledge should be assessed.[30] Because mathematical reasoning should be integrated across relevant content areas, professional development may be needed for teachers responsible for other content areas. Needs assessment tools are available to assist administrators, instructional coaches, or other teacher leaders in gathering this information (See Center on Instruction in Mathematics: http://www.centeroninstruction.org/identifying-professional-development-needs-in-mathematics-a-planning-tool-for-grades-3-7---second-edition).

A needs assessment can include teachers' self-reflection of their strengths and limitations, as well as an objective test of their knowledge and skills. Results from the needs assessment should be used to tailor professional development to support individual teacher's needs and to structure and inform the work of teacher professional learning communities.

PRACTICE 1 EXAMPLE APPLICATION: Steps for Conducting a Needs Assessment

1. Determine the purpose of the needs assessment and how the results will be used.
2. Determine who will administer the assessment, when, and how data will be analyzed and reported.
3. Identify the population of teachers who will take the needs assessment. Different populations might need differently designed needs assessments.
4. Identify what type of information is needed (e.g., self-perception, objective data, program evaluation).
5. Design the data collection instrument to capture the type of information needed.
6. Collect and organize data.
7. Analyze data based on the purpose of the needs assessment.
8. Use results of the needs assessment to inform design of professional development.

RESOURCE FOR EVALUATING QUALITY AND IMPACT OF PROFESSIONAL DEVELOPMENT

The Council for Chief State School Officers (CCSSO) analyzed evaluations of teacher professional development programs in mathematics and science. In conducting their review, they evaluated programs' effects on student outcomes, teacher content knowledge, and instructional practices. Additionally, they considered the quality of the research design and instruments used in making the evaluation. Reports published by objective organizations such as CCSSO provide a good resource for gathering information about the quality and potential impact of professional development programs.

Practice 2: *Select and implement high-quality professional development that acknowledges different teachers' needs.*

To effectively change teachers' practices, professional development opportunities should align with the standards and expectations proposed by Learning Forward (formerly the National Staff Development Council: http://www.learningforward.org) for staff development. Most importantly, professional development should be targeted to support individual teachers' needs. Because it takes time to build and grow professional knowledge, professional development should be delivered over time. Moreover, effective professional development should be situated within a collaborative environment, often within learning communities or by encouraging discourse among colleagues. All professional development opportunities should be evaluated for alignment with these expectations prior to implementation.

Practice 3: *Improve teachers' knowledge and understanding of making practice decisions based on research evidence and student data.*

A culture of using evidence to guide instructional decisions should focus on two types of activities.

First, student performance data should be systematically gathered at points before, during, and after instruction to guide instructional and programmatic decisions. Teachers and administrators should understand the types of data needed, how to collect and analyze data, and how to make decisions and communicate regarding the results.

Second, research- and evidence-based solutions should be integrated into teachers' and administrators' practices. Teachers and administrators should engage in ongoing professional development by reading research literature on mathematics instruction and interventions and by considering how these practices can be implemented within the local context. Moreover, administrators should cultivate a school climate that allows for experimentation and implementation of evidence-based practices in a supportive environment. Administrators and colleagues can work with teachers to support fidelity of implementation of research- and evidence-based practices.

Principle 9:

Discontinue using practices that are NOT associated with improved outcomes for students and teachers.

For students who are struggling in mathematics, it is essential that teachers employ practices that are supported by research. Fortunately, there are a number of resources available that describe and exemplify these practices. What follows are two recommended resources:

IES Practice Guide, *Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle grades.*[4] http://ies.ed.gov/ncee/wwc/pdf/practice_guides/rti_math_pg_042109.pdf

What Works Clearinghouse (Identifies research on programs as opposed to practices): http://ies.ed.gov/ncee/wwc/

Practice 1: *Examine the evidentiary bases of practices currently used in teaching mathematics and identify and eliminate practices that are contraindicated by existing evidence.*

There is a growing body of evidence regarding effective practices for teaching mathematics in middle grades. Research has been summarized on specific topics including instruction on fractions[6] and teaching struggling learners in mathematics.[4] While these documents represent important strides in knowledge about how best to provide instruction in middle grades mathematics for different types of students, even the document authors acknowledge that sufficient evidence to answer many important questions is lacking. For example, in the IES practice guide on struggling learners, it is recommended that instruction include motivational strategies for students who are receiving additional intervention in mathematics. However, there is little research available to support this recommendation. Rather, this recommendation comes primarily from the opinion of experts based on findings and theories in related areas or indirect evidence.

It is important that teachers and school leaders understand the current state of the evidence on mathematics instruction and how to use the evidence to examine their practices. Furthermore, it is very important that, in understanding the available evidence, teachers understand what is not known. It is also important for teachers to understand that in many cases, the available evidence focuses on elementary schools. However, in the absence of more substantial evidence, it is wise to consider this evidence and how it might relate to middle grades students.

For example, the use of manipulatives to teach mathematical concepts receives a lot of attention. Many online sources, print articles, and textbooks recommend the use of concrete objects to help students understand how mathematical ideas apply in the real world. These concrete objects (e.g., fraction bars, algebra tiles, abacus) are referred to as manipulatives because it is believed that students benefit from hands-on manipulation of these objects in developing their understanding of how mathematics works. The majority of research supporting manipulatives has been with students in the elementary grades; however, the research base related to the use of manipulatives to support mathematics learning in the middle grades is emerging.[31] Therefore, there are many questions that research has not yet answered, such as:

- How long do students need to manipulate these objects before they are ready to move to visual representations?
- If students already demonstrate understanding of a concept, do they need to go back to using the manipulatives to ensure they are not moving procedurally without sufficient foundational understanding?
- Which manipulatives help develop strong conceptual skill in each mathematical content area?

While awaiting more research across more mathematical content areas to address these and other questions, teachers should use their professional judgment on the necessity of hands-on activities and when they are most appropriately used in the classroom. Teachers should take this opportunity to conduct their own research using manipulatives. They can collect and analyze data to determine what is best for their students.

Practice 2: *Monitor student learning formally and informally and use trend data to determine whether and how to adjust current practices.*

Research summaries, such as those referenced above, provide answers to questions about the effectiveness of practices in particular contexts. Therefore, regular adjustments to instruction will be necessary. Decisions to make adjustments should be informed by multiple sources of student data. Decisions can be made at multiple levels: program, classroom, or student level. For example, based on grade-level data derived from state tests, a school may determine that a particular instructional program is not working for their students. A particular teacher may find that, based on periodic assessments of students' performance on a particular topic, additional support is needed for a group of students in the classroom in order for them to make sufficient progress.

Finally, progress monitoring with a single student or a small group of students may reveal that instructional adjustments need to be made to ensure that they learn a particular concept or skill or develop fluency with a particular procedure. For at-risk students, teachers should graph progress-monitoring data to determine whether these students are making adequate progress within their current mathematics program. Adequate progress may be determined by comparing the student's slope (i.e., trend in growth) to normative slope data or by comparing to a benchmark score.[5] If students do not demonstrate adequate progress, teachers should make instructional adaptations or changes. Continued progress monitoring will allow teachers to understand whether such adaptations and changes are beneficial for the student.

PRACTICE 2 EXAMPLE APPLICATION: Using Informal Data from Analyzing Student Work to Guide Instruction

A teacher has introduced equivalent fractions and has assigned a few practice examples for students to complete in groups. The group work requires students to determine whether two fractions are equivalent and to justify their answer. Each group successfully completes the assignment and appears to understand equivalent fractions. However, after reviewing their independent work the following day, the teacher notices a small group of students do not seem to understand that equivalent fractions, by definition, represent the same magnitude. Though the fractions have a different numerator and denominator, they are equivalent. Often, the students' equivalent fractions have different numerators but the same denominator. Based on this classroom data, the teacher makes the decision to use group work time to reteach the concept of equivalent fractions to this small group of students.

Conclusion

Mathematics is widely recognized as an important factor in furthering educational attainment and improving job market potential. A strong basis of mathematical education allows countries to develop the human capital needed to make scientific and technological advancements.[3] The principles and practices described in this section were derived from several sources including practice guides provided by the Institute of Education Sciences, meta-analyses in the research literature, and the report from the National Mathematics Advisory Panel (NMAP; 2008). Despite the importance of mathematics in everyday lives and to students' future academic success, many questions about how to teach mathematics effectively to the broadest range of students in the middle grades remain unanswered. Many of the principles and practices included in this section are supported by rigorous research. In instances such as Principles 1 and 2, where less rigorous research is available, principles and practices are based more on professional experience and clinical observations in schools. In addition, several areas that are currently considered important to mathematics learning have very little support in the research literature but warrant further investigation. For example, teacher knowledge of mathematics and how to teach mathematics are undoubtedly important to student achievement. Yet, only emerging evidence exists to support schools' efforts to provide professional development.

References:

Mathematics And Mathematics Interventions

1. The Center for Public Education. (2013). *Detail on mathematics graduation requirements from public high schools, by state*. Retrieved from http://www.centerforpubliceducation.org/Main-Menu/Policies/Understanding-the-Common-Core/Out-of-Sync-Many-Common-Core-states-have-yet-to-define-a-Common-Core-worthy-diploma/Detail-on-mathematics-graduation-requirements-from-public-high-schools-by-state.pdf

2. Star, J. R., Caronongan, P., Foegen, A., Furgeson, J., Keating, B., Larson, M. R., Lyskawa, J., McCallum, W. G., Porath, J., & Zbiek, R. M. (2015). *Teaching strategies for improving algebra knowledge in middle and high school students* (NCEE 2014-4333). Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/practiceguide.aspx?sid=20

3. National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education. Retrieved from http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf

4. Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., & Witzel, B. (2009). *Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle schools* (NCEE 2009-4060). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/practiceguide.aspx?sid=2

5. Fuchs, L. S., Fuchs, D., & Zumeta, R. O. (2008). A curricular-sampling approach to progress monitoring: Mathematics concepts and applications. *Assessment for Effective Intervention*, 33(4), 225-233. doi:10.1177/1534508407313484

6. Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE 2010-4039). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/PracticeGuide.aspx?sid=15

7. Gersten, R., & Chard, D. (1999). Number sense: Rethinking arithmetic instruction for students with mathematical disabilities. *Journal of Special Education*, 33(1), 18-28. doi:10.1177/002246699903300102

8. DeWolf, M., Bassok, M., & Holyoak, K. J. (2015). From rational numbers to algebra: Separable contributions of decimal magnitude and relational understanding of fractions. *Journal of Experimental Child Psychology*, 133, 72-84. doi:10.1016/j.jecp.2015.01.013

9. Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different countries. *Learning and Instruction*, 37, 5-13. doi:10.1016/j.learninstruc.2014.03.002

10. Rakes, C. R., Valentine, J. C., McGatha, M. B., & Ronau, R. N. (2010). Methods of instructional improvement in algebra: A systematic review and meta-analysis. *Review of Educational Research*, 80(3), 372-400. doi:10.3102/0034654310374880

11. Rittle-Johnson, B., Siegler, R. S., & Alibali, A. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93(2), 346-362. doi:10.1037/0033-0663.93.2.346

12. Ashlock, R. A. (2010*). Error patterns in computation: Using error patterns to help each student learn* (10th ed.). Boston, MA: Allyn & Bacon.

13. Ball, D. L., Ferrini-Mundy, J., Kilpatrick, J., Milgram, R. J., Schmid, W., & Schaar, R. (2005). Reaching for common ground in K-12 mathematics education. *Notices of the AMS*, 52(9), 1055–58.

14. Montague, M., Krawec, J., Enders, C., & Dietz, S. (2014). The effects of cognitive strategy instruction on math problem solving of middle-school students of varying ability. *Journal of Educational Psychology*, 106(2), 469-481. doi:10.1037/a0035176

15. Slavin, R. E., Lake, C., & Groff, C. (2009). Effective programs in middle and high school mathematics: A best-evidence synthesis. *Review of Educational Research*, 79(2), 839-911. doi:10.3102/0034654308330968

16. Kamps, D. M., Greenwood, C., Arreaga-Mayer, C., Veerkamp, M. B., Utley, C., Tapia, Y., … Bannister, H. (2008). The efficacy of classwide peer tutoring in middle schools. *Education and Treatment of Children*, 31(2), 119-152.

17. Jitendra, A. K., & Star, J. R. (2011). Meeting the needs of students with learning disabilities in inclusive mathematics classrooms: The role of schema-based instruction on mathematical problem-solving. *Theory Into Practice*, 50(1), 12-19. doi:10.1080/00405841.2011.534912

18. Pashler, H., Bain, P., Bottge, B., Graesser, A., Koedinger, K., McDaniel, M., & Metcalf, J. (2007). *Organizing instruction and study to improve student learning: IES practice guide* (NCER 2007-2004). Washington, DC: National Center for Education Research, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/PracticeGuide.aspx?sid=1

19. Rohrer, D., & Taylor, K. (2006). The effects of overlearning and distributed practice on the retention of mathematics knowledge. *Applied Cognitive Psychology*, 20(9), 1209-1224. doi:10.1002/acp.1266

20. Cepeda, N. J., Pashler, H., Vul, E., Wixted, J. T., & Rohrer, D. (2006). Distributed practice in verbal recall tasks: A review and quantitative synthesis. *Psychological Bulletin*, 132(3), 354-380. doi:10.1037/0033-2909.132.3.354

21. Jitendra, A. K., Petersen-Brown, S., Lein, A. E., Zaslofsky, A. F., Kunkel, A. K., Jung, P. G., & Egan, A. M. (2015). Teaching mathematical word problem solving: The quality of evidence for strategy instruction priming the problem structure. *Journal of Learning Disabilities*, 48(1), 51-72. doi:10.1177/0022219413487408

22. Woodward, J., Beckmann, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2012). *Improving mathematical problem solving in grades 4 through 8: A practice guide* (NCEE 2012-4055). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/practiceguide.aspx?sid=16

23. Krawec, J., Huang, J., Montague, M., Kressler, B., & De Alba, A. M. (2012). The effects of cognitive strategy instruction on knowledge of math problem-solving processes of middle school students with learning disabilities. *Learning Disability Quarterly*, 36(2), 80-92. doi:10.1177/0731948712463368

24. Van Garderen, D., Scheuermann, A., & Jackson, C. (2012). Examining how students with diverse abilities use diagrams to solve mathematics word problems. *Learning Disability Quarterly*, 36(3), 145-160. doi:10.1177/0731948712438558

25. Van Garderen, D., Scheuermann, A., & Jackson, C. (2012). Developing representational ability in mathematics for students with learning disabilities: A content analysis of grades 6 and 7 textbooks. *Learning Disability Quarterly*, 35(1), 24-38. doi:10.1177/0731948711429726

26. Stylianou, D. A. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Mathematics Teacher Education*, 13, 325-343. doi:10.1007/s10857-010-9143-y

27. Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(1), 14-46.

28. Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.

29. Wu, H. (2011). *The mathematics K-12 teachers need to know*. Department paper retrieved from http://math.berkeley.edu/~wu/Schoolmathematics1.pdf

30. Sample McMeeking, L. B., Orsi, R., & Cobb, R. B. (2012). Effects of a teacher professional development program on the mathematics achievement of middle school students. *Journal for Research in Mathematics Education*, 43(2), 159-181.

31. Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2012). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, 105(2), 380-400. doi:10.1037/a0031084